Isomonodromic tau-function of Hurwitz Frobenius manifolds and its applications

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Abstract. In this work we find the isomonodromic (Jimbo-Miwa) tau-function corresponding to Frobenius manifold structures on Hurwitz spaces. We discuss several applications of this result. First, we get an explicit expression for the G-function (solution of Getzler's equation) of the Hurwitz Frobenius manifolds. Second, in terms of this tau-function we compute the genus one correction to the free energy of hermitian two-matrix model. Third, we find the Jimbo-Miwa tau-function of an arbitrary Riemann-Hilbert problem with quasi-permutation monodromy matrices. Finally, we get a new expression (analog of genus one Ray-Singer formula) for the determinant of Laplace operator in the Poincaré metric on Riemann surfaces of an arbitrary genus.

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1 Introduction

The Hurwitz space $H_{g,N}$ is the space of equivalence classes of pairs $(\mathcal{L},\,\pi)$, where $\mathcal{L}$ is a compact Riemann surface of genus g and π is a meromorphic function of degree N. The Hurwitz space is stratified according to multiplicities of poles and critical points of function π (see [25, 13]); in this paper we shall mainly work within the generic stratum $H_{g,N}(1,\dots,1)$, for which all critical points and poles of function π are simple. Denote the critical points of function π by $P_1,\dots P_M$ ($M=2N+2g-2$ according to the Riemann-Hurwitz formula); the critical values $\lambda_m=\pi(P_m)$ can be used as (local) coordinates on $H_{g,N}(1,\dots,1)$. The function π defines a realization of the Riemann surface $\mathcal{L}$ as an N-sheeted branched covering of $\mathbb{C}P^1$ with ramification points $P_1,\dots,P_M$ and branch points $\lambda_m=\pi(P_m)$; enumerate the points at infinity of the branched covering in some order and denote them by $\infty_1,\dots,\infty_N$. In a neighbourhood of the ramification point P_m the local coordinate is chosen to be $x_m(P)=\sqrt{\pi(P)-\lambda_m}$, $m=1,\dots,M$; in a neighbourhood of any point ∞_n the local parameter is $x_{M+n}(P)=1/\pi(P)$, $n=1,\dots,N$.

Fix a canonical basis of cycles $(a_\alpha,\,b_\alpha)$ on $\mathcal{L}$ and introduce the prime-form $E(P,Q)$ on $\mathcal{L}$ and canonical meromorphic bidifferential

$$W(P,Q)=d_Pd_Q\ln E(P,Q) \tag{1.1}$$

The bidifferential W has the second order pole at $Q=P$ with the following local behaviour:

$$\frac{W(P,Q)}{dx(P)dx(Q)} = \frac{1}{(x(P)-x(Q))^2} + \frac{1}{6}S_B(x(P)) + o(1)\;,$$

where $x(P)$ is a local coordinate; $S_B(x(P))$ is the Bergman projective connection.

The central object of this paper is the function $\tau(\lambda_1,\dots,\lambda_M)$ (the "tau-function") defined by the following system of equations:

$$\frac{\partial}{\partial\lambda_m}\ln\tau = -\frac{1}{12}S_B(x_m)|_{x_m=0}\;,\qquad m=1,\dots,M\;; \tag{1.2}$$

compatibility of this system can be obtained as a simple corollary of the Rauch variational formulas [18]. In global terms, τ is a horizontal holomorphic section of the flat holomorphic line bundle $\mathcal{T}_B$ (see [18]) over the space $\widehat{H_{g,N}}(1,\dots,1)$, which covers $H_{g,N}(1,\dots,1)$, and consists of pairs (weakly marked Riemann surface $\mathcal{L}$; meromorphic function π with simple poles and critical points). This covering space appears due to dependence of the bidifferential W (and, therefore, the Bergman projective connection) on the choice of homology basis on $\mathcal{L}$.

In the Frobenius manifolds theory [4], apart from the prepotential (solutions of WDVV equations), an important role is played by the so-called G-function, which is the genus one free energy, corresponding to a given Frobenius manifold (the prepotential itself equals to the planar limit of the free energy). It was conjectured by Givental [14] and proved by Dubrovin-Zhang [6] that the G-function can be expressed in terms of Jimbo-Miwa tau-function of the isomonodromic problem corresponding to a given Frobenius manifold.

In [4] Frobenius manifold structures were found on an arbitrary Hurwitz space; so far this is, probably, one of most well-understood classes of Frobenius manifolds (alternative structures of Frobenius manifolds on Hurwitz spaces were recently found in [27, 28]). As it was recently proved in [19], the definition of isomonodromic tau-function $\tau(\lambda_1,\dots,\lambda_M)$ of Hurwitz Frobenius manifolds from [4] is equivalent to (1.2).

The same tau-function (1.2) appears as one of two multipliers in the Jimbo-Miwa tau-function corresponding to another class of Riemann-Hilbert problems - the Riemann-Hilbert problems with quasi-permutation monodromy matrices [22].

In [18] it was also revealed the role of the function τ in the problem of holomorphic factorization of the determinant of the Laplacian on Riemann surfaces: namely, up to a factor involving an appropriate regularized Dirichlet integral and the matrix of b-periods of a Riemann surface, the determinant of Laplace operator (in the Poincaré metric) acting in the trivial line bundle over Riemann surface is given by $|\tau|^2$.

Another important area where the same tau-function appeared recently is the large N limit of Hermitian two-matrix model [9]; in this paper it was realized that the subleading correction to the free energy of such models formally almost coincides with the G-function of Hurwitz Frobenius manifolds. In particular, the isomonodromic tau-function (1.2) is the most non-trivial ingredient of this subleading correction.

For $N=2$ and arbitrary g the Riemann surface $\mathcal{L}$ is hyperelliptic, and can be defined by equation $w^2=\prod_{m=1}^{2g+2}(\lambda-\lambda_m)$. In this case $\tau=\det\mathbf{A}\prod_{m\neq n}^{M}(\lambda_m-\lambda_n)^{1/4}$ [16], where $\mathbf{A}$ is the matrix of a-periods of non-normalized holomorphic differentials $\lambda^{\alpha-1}d\lambda/w\;,\;\alpha=1,\dots,g$. In other simple case, when $g=0,1$ and N is arbitrary,the function τ was found in [18]; this result allowed to compute the G-function of Frobenius manifold related to the extended affine Weyl group $\tilde{W}(A_{N-1})$ (originally found

in [29]) and the G-function of Frobenius manifold related to the Jacobi group $J(A_{N-1})$ (conjectured in [29]).

The goal of this paper is to compute the tau-function of the isomonodromy problem corresponding to Frobenius structures on an arbitrary Hurwitz space $H_{g,N}(1,\dots,1)$.

Consider the divisor $\mathcal{D}$ of the differential $d\pi$: $\mathcal{D}=\sum_{k=1}^{M+N} d_k D_k$, where $D_m=P_m$, $d_m=1$ for $m=1,\dots,M$ and $D_{M+n}=\infty_n$, $d_{M+n}=-2$ for $n=1,\dots,N$. Here and below, if the argument of a differential coincides with a point D_k of divisor $\mathcal{D}$, we evaluate this differential at this point with respect to local parameter x_k. In particular, for the prime form we shall use the following conventions:

$$E(D_k,D_l):=E(P,Q)\sqrt{dx_k(P)}\sqrt{dx_l(Q)}|_{P=D_k,\,Q=D_l}\ , \tag{1.3}$$

for $k,l=1,\dots,M+N$. The next notation corresponds to prime-forms, evaluated at points of divisor $\mathcal{D}$ with respect to only one argument:

$$E(P,D_k):=E(P,Q)\sqrt{dx_k(Q)}|_{Q=D_k}\ , \tag{1.4}$$

$k=1,\dots,M+N$; in contrast to $E(D_k,D_l)$, which are just scalars, $E(P,D_k)$ are $-1/2$-forms with respect to P.

Denote by $v_1,\dots,v_g$ the normalized ($\oint_{a_\alpha} v_\beta=\delta_{\alpha\beta}$) holomorphic differentials on $\mathcal{L}$; $\mathbf{B}_{\alpha\beta}=\oint_{b_\alpha} v_\beta$ is the corresponding matrix of b-periods; $\Theta(z|\mathbf{B})$ is the theta-function. Let us dissect the Riemann surface $\mathcal{L}$ along its basic cycles to get its fundamental polygon $\hat{\mathcal{L}}$; choose some initial point $P\in\hat{\mathcal{L}}$ and introduce the corresponding vector of Riemann constants

$$K_\alpha^P=\frac{1}{2}+\frac{1}{2}\mathbf{B}_{\alpha\alpha}-\sum_{\beta\neq\alpha}\oint_{a_\beta}(v_\beta(Q)\int_P^Q v_\alpha)\ ;\quad \alpha=1,\dots,g \tag{1.5}$$

and the Abel map $[\mathcal{A}_P]_\alpha(Q)=\int_P^Q v_\alpha$, computed along path which does not intersect $\partial\hat{\mathcal{L}}$.

The following theorem, together with its applications, is the main result of this paper

Theorem 1 *Assume that the fundamental domain $\hat{\mathcal{L}}$ is chosen in such a way that*

$$\mathcal{A}(\mathcal{D})+2K^P=0. \tag{1.6}$$

The isomonodromic tau-function (1.2) of a Frobenius manifold associated to the Hurwitz space $H_{g,N}(1,\dots,1)$ is given by the following expression:

$$\tau=\mathcal{F}^{2/3}\prod_{k,l=1\ k<l}^{M+N}[E(D_k,D_l)]^{\frac{d_k d_l}{6}} \tag{1.7}$$

where the quantity $\mathcal{F}$ defined by

$$\mathcal{F}=\frac{[d\pi(P)]^{\frac{g-1}{2}}}{\mathcal{W}(P)}\left\{\prod_{k=1}^{M+N}[E(P,D_k)]^{\frac{(1-g)d_k}{2}}\right\}\sum_{\alpha_1,\dots,\alpha_g=1}^{g}\frac{\partial^g\Theta(K^P)}{\partial z_{\alpha_1}\dots\partial z_{\alpha_g}}v_{\alpha_1}(P)\dots v_{\alpha_g}(P) \tag{1.8}$$

is independent of the point $P\in\mathcal{L}$. Here Θ is the theta-function of $\mathcal{L}$; integer vector $\mathbf{r}$ is defined by (1.6); $\langle\ ,\ \rangle$ is the scalar product in $\mathbb{C}^g$, ($\langle x,y\rangle=\sum_{\alpha=1}^g x_\alpha y_\alpha$);

$$\mathcal{W}(P):=\det_{1\le\alpha,\beta\le g}||v_\beta^{(\alpha-1)}(P)|| \tag{1.9}$$

denotes the Wronskian determinant of holomorphic differentials at the point P.

The proof of this theorem is contained in Section 2.

In section 3 we discuss applications of the formula (1.7). First, in Sect.3.1 we show how to find the G-function of Frobenius manifolds from [4] corresponding to Hurwitz spaces; the resulting formula looks as follows:

$$G = -\frac{1}{2}\ln\tau - \frac{1}{48}\sum_{m=1}^{M} \ln \operatorname{Res}_{P_m}\frac{\varphi^2}{d\lambda}\ , \tag{1.10}$$

where φ is a primary differential defining the Frobenius manifold [1]. In Section 3.2 we use the tau-function (1.7) to compute the genus one correction to the free energy of hermitian two-matrix matrix model. In section 3.3 the formula (1.7) is used to get a new expression (valid up to a constant independent of moduli of the Riemann surface) for the determinant of the Laplacian on Riemann surface $\mathcal{L}$ in Poincaré metric. A formula for $\det\Delta$ in Arakelov metric was proved by Fay [12]; combining this formula with Polyakov's formula relating determinants of Laplacians in different conformal metrics on the same Riemann surface, one can get an expression for $\det\Delta$ in the Poincaré metric. The expression we derive here is different, and is given by the modulus square of the tau-function (1.7) multiplied by the exponent of an appropriate Dirichlet integral. In section 3.4 we show how to apply the formula (1.7) to find the Jimbo-Miwa tau-function of another class of Riemann-Hilbert problems - the ones with arbitrary quasi-permutation monodromy matrices [22].

This paper is based on the authors' preprint [17].

2 Proof of the main theorem

2.1 Variational formulas on the spaces of branched coverings

Here we establish the formulae describing the variations of basic holomorphic objects (holomorphic differentials, the canonical bidifferential, the prime-form, the vector of Riemann constants, etc) on the Riemann surface $\mathcal{L}$ under the variation of a critical value of the map $\pi\ :\ \mathcal{L}\to\mathbb{C}P^1$.

With a slight abuse of terminology we denote the branched covering of the Riemann sphere defined by the function π on the Riemann surface $\mathcal{L}$ by the same letter $\mathcal{L}$; the coordinate on the covered Riemann sphere will be denoted by λ. The zeros of $d\pi$ are the ramification point of the branched covering $\mathcal{L}$; the local parameter in a neighbourhood of P_m is $x_m = \sqrt{\lambda-\lambda_m}$, according to the notations from introduction.

First, we recall the properties of the prime form $E(P,Q)$ (see [11, 12]), which is an antisymmetric $-1/2$-differential with respect to both P and Q:

- Under tracing of Q along the cycle a_α the prime-form remains invariant; under the tracing along b_α it gains the factor

$$\exp(-\pi i\mathbf{B}_{\alpha\alpha} - 2\pi i\int_P^Q v_\alpha)\ . \tag{2.1}$$

[1] The tau-function τ is defined according to original formula of Jimbo-Miwa [15]; the isomonodromic tau-function τ_I defined in [6] is related to τ as follows: $\tau_I = \tau^{-1/2}$ (we thank V.Shramchenko for this observation). Here we prefer the convention of [15], since it is this definition which guarantees the holomorphy of the tau-function in general case.

- The prime-form can be expressed in terms of the canonical meromorphic bidifferential $W(P,Q)$ as follows:
(2.2)
$$E^2(P,Q)dx(P)dy(Q) = \lim_{P_0\to P, Q_0\to Q}(x(P_0)-x(P))(y(Q)-y(Q_0))\exp\left(-\int_{P_0}^{Q_0}\int_P^Q W(\,\cdot\,,\,\cdot\,)\right),$$

- At the diagonal $Q=P$ the prime-form has the first order zero; the following asymptotics holds:
$$E(x(P),x(Q))\sqrt{dx(P)}\sqrt{dx(Q)} =$$
$$(x(Q)-x(P))\left(1-\frac{1}{12}S_B(x(P))(x(Q)-x(P))^2+O((x(Q)-x(P))^3\right), \tag{2.3}$$
as $Q\to P$, where S_B is the Bergman projective connection and $x(P)$ is an arbitrary local parameter.

Now for any two points $P,Q\in\hat{\mathcal{L}}$ we define
$$\mathbf{s}(P,Q) := \exp\left\{-\sum_{\alpha=1}^{g}\oint_{a_\alpha} v_\alpha(R)\ln\frac{E(R,P)}{E(R,Q)}\right\} \tag{2.4}$$
This object is a (multi-valued) non-vanishing holomorphic $g/2$-differential on $\hat{\mathcal{L}}$ with respect to P and non-vanishing holomorphic $-g/2$-differential with respect to Q. Under tracing along the cycles a_α and b_α it gains the multipliers 1 and $\exp[(g-1)\pi i\mathbf{B}_{\alpha\alpha}+2\pi i K_\alpha^P]$ with respect to P and the multipliers 1 and $\exp[(1-g)\pi i\mathbf{B}_{\alpha\alpha}-2\pi i K_\alpha^Q]$ with respect to Q.

As in the case of the prime form, if one of the arguments coincides with a point of $\mathcal{D}$, we evaluate $\mathbf{s}$ in the corresponding local parameter:
$$\mathbf{s}(D_k,Q) := \mathbf{s}(P,Q)(dx_k(P))^{-g/2}|_{P=D_k} \tag{2.5}$$
where $k=1,\dots,M+N$. For arbitrary two points $P,Q\notin D$ we introduce the following notation:
$$\sigma(P,Q) := \mathbf{s}(P,Q)(d\pi(P))^{-g/2}(d\pi(Q))^{g/2} \tag{2.6}$$

Let us fix two points P_0, $Q_0\in\mathcal{L}$ (for convenience in the sequel we assume that these points do not coincide with points of $\mathcal{D}$), and introduce another object which plays an important role below, the following (non-single-valued) holomorphic 1-differential on $\hat{\mathcal{L}}$:
$$\omega(P) = \mathbf{s}^2(P,Q_0)E(P,P_0)^{2g-2}(d\pi(Q_0))^g(d\pi(P_0))^{g-1}\ ; \tag{2.7}$$
in agreement with the previous notations, $\omega(D_k) := \frac{\omega(P)}{dx_k(P)}|_{P=D_k}$. The differential $\omega(P)$ has multipliers 1 and $\exp(4\pi i K_\alpha^{P_0})$ along the basic cycles a_α and b_α respectively. The only zero of the 1-form ω on $\hat{\mathcal{L}}$ is P_0; its multiplicity equals $2g-2$.

Consider the following Schwarzian derivative (which depends on the chosen point P_0, but is obviously independent of the point Q_0 from (2.7))
$$S_{Fay}^{P_0}(x(P)) := \left\{\int^P\omega(P),\ x(P)\right\}\ , \tag{2.8}$$

where $x(P)$ is a local coordinate on $\mathcal{L}$; $S_{Fay}^{P_0}$ is a projective connection (see e.g. [30]) on $\mathcal{L}$; this object was introduced and exploited by Fay [12] in a different form (and without mentioning that it is a projective connection).

Introduce also the following holomorphic non-single-valued $g(1-g)/2$-differential on $\hat{\mathcal{L}}$ which has multipliers 1 and $\exp\{-\pi i(g-1)^2\mathbf{B}_{\alpha\alpha} - 2\pi i(g-1)K_\alpha^P\}$ along basic cycles a_α and b_α, respectively:

$$\mathcal{C}(P) = \frac{1}{\mathcal{W}[v_1,\dots,v_g](P)} \sum_{\alpha_1,\dots,\alpha_g=1}^{g} \frac{\partial^g\Theta(K^P)}{\partial z_{\alpha_1}\dots\partial z_{\alpha_g}} v_{\alpha_1}\dots v_{\alpha_g}(P)\;, \tag{2.9}$$

where $\mathcal{W}(P)$ is the Wronskian (1.9) from the introduction.

The differential $\mathcal{C}$ is an essential ingredient of the Mumford measure on the moduli space of Riemann surfaces of given genus [12]. For $g>1$ the multiplicative differential $\mathbf{s}$ (2.4) is expressed in terms of $\mathcal{C}$ as follows [12]:

$$\mathbf{s}(P,Q) = \left(\frac{\mathcal{C}(P)}{\mathcal{C}(Q)}\right)^{1/(1-g)} \tag{2.10}$$

According to Corollary 1.4 from[12], $\mathcal{C}(P)$ does not have any zeros. Moreover, this object admits the following alternative representation:

$$\mathcal{C}(P) = \frac{\Theta(\sum_{\alpha=1}^{g-1}\mathcal{A}_P(R_\alpha) + \mathcal{A}_Q(R_g) + K^P)\prod_{\alpha<\beta}E(R_\alpha,R_\beta)\prod_{\alpha=1}^{g}\mathbf{s}(R_\alpha,P)}{\prod_{\alpha=1}^{g}E(Q,R_\alpha)\det||v_\alpha(R_\beta)||_{\alpha,\beta=1}^{g}\mathbf{s}(Q,P)}. \tag{2.11}$$

where $Q, R_1,\dots,R_g \in \mathcal{L}$ are arbitrary points of $\mathcal{L}$.

The following Theorem describes the behaviour of the basic holomorphic differentials v_α, the matrix $\mathbf{B}$ of b-periods, the canonical bidifferential $W(P,Q)$, the prime form $E(P,Q)$, the vector of Riemann constants K^P, and the multiplicative differentials $\mathbf{s}(P,Q)$ and $\mathcal{C}(P)$ under variations of the critical values λ_m.

From now on we use the notation

$$\partial_m T(P_m) := \frac{dT(x_m)}{dx_m}\Big|_{x_m=0}$$

for the derivative of a tensor $T(x_m)(dx_m)^r$ of a (possibly fractional) weight r at the critical point P_m calculated with respect to the local parameter x_m.

Theorem 2 *Let the coordinates $\lambda(P)=\pi(P)$ and $\lambda(Q)=\pi(Q)$ of the points P and Q do not change when the covering $\pi:\mathcal{L}\to\mathbb{C}P^1$ deforms. Under the convention that all the tensor objects with arguments P, Q and Q_0 are calculated in the local parameter λ lifted from the base $\mathbb{C}P^1$ of the covering π and all the tensor objects with argument P_m are calculated in the local parameter $x_m=\sqrt{\lambda-\lambda_m}$, the following variational formulae hold*

$$\frac{\partial v_\alpha(P)}{\partial\lambda_m} = \frac{1}{2}W(P,P_m)v_\alpha(P_m), \tag{2.12}$$

$$\frac{\partial \mathbf{B}_{\alpha\beta}}{\partial\lambda_m} = \pi i v_\alpha(P_m)v_\beta(P_m), \tag{2.13}$$

$$\frac{\partial W(P,Q)}{\partial \lambda_m} = \frac{1}{2} W(P,P_m) W(P_m,Q), \tag{2.14}$$

$$\frac{\partial E(P,Q)}{\partial \lambda_m} = -\frac{1}{4}\left[\partial_m \ln \frac{E(P,P_m)}{E(Q,P_m)}\right]^2, \tag{2.15}$$

$$\frac{\partial K_\alpha^P}{\partial \lambda_m} = \frac{1}{2} v_\alpha(P_k) \partial_m \ln\left(\mathbf{s}(P_m,Q_0) E(P_m,P)^{g-1}\right) - \frac{1}{4}\partial_m v_\alpha(P_m)\ , \tag{2.16}$$

$$\frac{\partial \sigma(P,Q)}{\partial \lambda_m} = \frac{1}{4}\left\{\partial_m \ln \frac{E(P_m,P)}{E(P_m,Q)}\right\}\left\{\partial_m \ln\left[\mathbf{s}(P_m,Q_0)^2 E(P_m,P)^{g-1} E(P_m,Q)^{g-1}\right]\right\}$$

$$-\frac{1}{4}\partial_m^2 \ln \frac{E(P_m,P)}{E(P_m,Q)}\ , \tag{2.17}$$

$$\frac{\partial \mathcal{C}(P)}{\partial \lambda_m} = -\frac{1}{8}(S_B - S_{Fay}^P)(P_m)\ , \tag{2.18}$$

where the expressions in the right hand sides of (2.16), (2.17) and (2.18) are Q_0-independent.

Remark 1 Formally, the expressions (2.12-2.18) are complete analogs of variational formulas (3.21), (3.22), (3.24) and (3.25) from [12]. However, Th. 2 cannot be obtained as a straightforward consequence of the formulae from [12]. In the scheme of moduli deformation used in [12] the C^∞-surface $\mathcal{L}$ is fixed and the systems of coordinates defining the complex structures on $\mathcal{L}$ vary, while we use the branch points as the local coordinates on the moduli space. Although it is not too difficult to establish a direct correspondence between the two deformation schemes for the objects which don't depend on a point of the Riemann surface (see for example [18]), for the point-dependent objects (like all the objects listed in the theorem except the matrix of b-periods) it is much less trivial, since the fixing of the argument in the two schemes is essentially different. Therefore, we prove the theorem independently; formally the proof looks very similar to [12].

Proof. An elementary proof of formulae (2.12-2.14) can be found in [18]. As in [12], formula (2.15) immediately follows from (2.14) and (2.2). Let us prove (2.16).

One may assume that the projections of a- and b-cycles on λ-plane do not move when the covering deforms. Varying the right hand side of (1.5) via (2.13) and (2.12) and taking into account (1.1), we get

$$\partial_{\lambda_m} K_\alpha^P = \frac{\pi i}{2} v_\alpha(P_m)^2 - \sum_{\beta\neq\alpha} \oint_{a_\beta} \frac{1}{2}\left\{\partial_\lambda \partial_m \ln E(\lambda,P_m) v_\beta(P_m)\int_P^\lambda v_\alpha\right\} d\lambda$$

$$-\oint_{a_\beta} v_\beta(\lambda)\int_P^\lambda \left(\frac{1}{2}\partial_{\lambda'}\partial_m \ln E(\lambda',P_m) v_\alpha(P_m)\right) d\lambda' =$$

$$\frac{\pi i}{2}v_\alpha(P_m)^2 - \frac{1}{2}\sum_{\beta\neq\alpha} v_\beta(P_m)\oint_{a_\beta}\left\{(\partial_\lambda\partial_m \ln E(\lambda,P_m))\int_P^\lambda v_\alpha\right\}d\lambda$$

$$-\frac{v_\alpha(P_m)}{2}\sum_{\beta\neq\alpha}\oint_{a_\beta} v_\beta(\lambda)\partial_m \ln\frac{E(\lambda,P_m)}{E(P,P_m)} =$$

$$\frac{\pi i}{2}v_\alpha(P_m)^2 + \frac{g-1}{2}v_\alpha(P_m)\partial_m \ln E(P,P_m) - \frac{1}{2}\sum_{\beta\neq\alpha} v_\beta(P_m)\oint_{a_\beta}\left\{\partial_\lambda\partial_m E(\lambda,P_m)\int_P^\lambda v_\alpha\right\}d\lambda$$

$$-\frac{v_\alpha(P_m)}{2}\sum_{\beta\neq\alpha}\oint_{a_\beta} v_\beta(\lambda)\partial_m \ln E(\lambda,P_m) =$$

$$= \frac{\pi i}{2}v_\alpha(P_m)^2 + \frac{v_\alpha(P_m)}{2}\partial_m \ln E^{g-1}(P,P_m) - \frac{1}{2}\sum_{\beta\neq\alpha} v_\beta(P_m)\oint_{a_\beta}\left\{\partial_\lambda\partial_m E(\lambda,P_m)\int_P^\lambda v_\alpha\right\}d\lambda$$

$$+\frac{v_\alpha(P_m)}{2}\partial_m \ln \mathbf{s}(P_m,Q_0) + \frac{v_\alpha(P_m)}{2}\oint_{a_\alpha} v_\alpha(\lambda)\partial_m \ln E(P_m,\lambda) =$$

$$\frac{v_\alpha(P_m)}{2}\partial_m \ln \mathbf{s}(P_m,Q_0)E^{g-1}(P,P_m) + \frac{\pi i}{2}v_\alpha(P_m)^2 + \frac{1}{2}\sum_{\beta=1}^{g} v_\beta(P_m)\oint_{a_\beta}\{\partial_m \ln E(\lambda,P_m)\}\,v_\alpha(\lambda)\,, \tag{2.19}$$

where the last equality is obtained via integration by parts (which is possible since the prime form has no twists along the a-cycles). Following Fay ([12]), we notice that, due to (2.1), the sum of the last two terms in the latter expression coincides with the following integral over the boundary of the fundamental polygon $\hat{\mathcal{L}}$

$$-\frac{1}{8\pi i}\oint_{\partial\hat{\mathcal{L}}} v_\alpha(\lambda)\left(\partial_m \ln E(\lambda,P_m)\right)^2\,. \tag{2.20}$$

From asymptotics (2.3) and the Cauchy formula it follows that integral (2.20) coincides with

$$-\frac{1}{4}v'_\alpha(x_m)\Big|_{x_m=0} \equiv -\frac{1}{4}\partial_m v_\alpha(P_m)\,,$$

which gives equation (2.16).

Let us prove (2.17). Due to (2.12) and (2.15), we have

$$\partial_{\lambda_m}\ln \mathbf{s}(P,Q) = -\partial_{\lambda_m}\sum_{\beta=1}^{g}\oint_{a_\beta} v_\beta(\lambda)\ln\frac{E(\lambda,P)}{E(\lambda,Q)} = -\frac{1}{2}\sum_{\beta=1}^{g}\oint_{a_\beta}[\partial_\lambda\partial_m \ln E(P_m,\lambda)]v_\beta(P_m)\ln\frac{E(\lambda,P)}{E(\lambda,Q)}\,d\lambda$$

$$+\frac{1}{4}\sum_{\beta=1}^{g}\oint_{a_\beta} v_\beta(\lambda)\left\{\left(\partial_m \ln\frac{E(\lambda,P_m)}{E(P,P_m)}\right)^2 - \left(\partial_m \ln\frac{E(\lambda,P_m)}{E(Q,P_m)}\right)^2\right\} := \Sigma_1+\Sigma_2. \tag{2.21}$$

To simplify the first sum in (2.21) we integrate it by parts, rewrite the resulting expression as the integral over the boundary of the fundamental polygon and apply the Cauchy theorem:

$$\Sigma_1 = \frac{1}{2}\sum_{\beta=1}^{g}\oint_{a_\beta} v_\beta(P_m)\partial_m \ln E(P_m,\lambda)\partial_\lambda \ln\frac{E(\lambda,P)}{E(\lambda,Q)}\,d\lambda = -\frac{1}{8\pi i}\oint_{\partial\hat{\mathcal{L}}}(\partial_m \ln E(P_m,\lambda))^2\partial_\lambda \ln\frac{E(\lambda,P)}{E(\lambda,Q)}\,d\lambda =$$

$$-\frac{1}{4}\left[\partial_m^2 \ln \frac{E(P_m,P)}{E(P_m,Q)} + (\partial_m \ln E(P,P_m))^2 - (\partial_m \ln E(Q,P_m))^2\right]. \tag{2.22}$$

Here we used the fact that the function

$$\lambda \mapsto \partial_\lambda \ln \frac{E(\lambda,P)}{E(\lambda,Q)}$$

is single-valued on $\mathcal{L}$ and that

$$\oint_{a_\beta} \partial_\lambda \ln \frac{E(\lambda,P)}{E(\lambda,Q)}\, d\lambda = 0,$$

due to the single-valuedness of the prime form along the a-cycles.

The second sum in (2.21) can be rewritten as

$$\Sigma_2 = \frac{1}{4}\sum_{\beta=1}^{g} \oint_{a_\beta} v_\beta(\lambda)\left\{2\partial_m \ln E(\lambda,P_m)\partial_m \ln \frac{E(Q,P_m)}{E(P,P_m)} + \partial_m \ln(E(P,P_m)E(Q,P_m))\partial_m \ln \frac{E(P,P_m)}{E(Q,P_m)}\right\} =$$

$$-\frac{1}{2}\partial_m \ln \frac{E(P,P_m)}{E(Q,P_m)} \sum_{\beta=1}^{g}\left\{\partial_m \oint_{a_\beta} v_\beta(\lambda) \ln \frac{E(\lambda,P_m)}{E(\lambda,Q_0)}\right\} + \frac{g}{4}\partial_m \ln(E(P,P_m)E(Q,P_m))\partial_m \ln \frac{E(P,P_m)}{E(Q,P_m)}$$

$$= \frac{1}{4}\partial_m \ln \frac{E(P,P_m)}{E(Q,P_m)} \partial_m \ln\left[\mathbf{s}^2(P_m,Q_0)E^g(P,P_m)E^g(Q,P_m)\right]. \tag{2.23}$$

Relation (2.17) follows from (2.22) and (2.23).

Now we are in a position to prove the main statement (2.18) of the theorem.

Let us first rewrite the definition of Fay's projective connection (2.8) in a local parameter ζ as follows:

$$S^P_{Fay}(\zeta) = 2\partial^2_{\zeta\zeta} \ln[\mathbf{s}(\zeta,Q_0)E(\zeta,P)^{g-1}(d\zeta)^{-1/2}] - 2(\partial_\zeta \ln[\mathbf{s}(\zeta,Q_0)E(\zeta,P)^{g-1}(d\zeta)^{-1/2}])^2. \tag{2.24}$$

Similarly to ([12]), to prove (2.18) we are to vary the logarithm of the right hand side of expression (2.11) and pass to the limit $R_1,\dots,R_g \to P$, and then $Q \to P_m$.

In what follows all the tensor objects with arguments $P, Q, R_1, \dots, R_g$ are calculated in the local parameter λ and, as before, the appearance of the argument P_m means means that the corresponding tensor is calculated in the local parameter x_m at the point $x_m = 0$.

The next lemma describes the variation of the determinant $\det||v_\alpha(R_\beta)||$ from the denominator of expression (2.11).

Lemma 1 *Assume that none of the points $R_1, \dots R_g$ coincide with the ramification points $\{P_m\}$, and the projections of the points $\{R_\alpha\}$ on λ-plane don't depend on $\{\lambda_m\}$. Then the following variational formula holds*

$$\lim_{R_1,\dots,R_g \to P} \frac{\partial \ln \det ||v_\alpha(R_\beta)||}{\partial \lambda_m} = -\frac{1}{2}\sum_{\alpha,\beta=1}^{g} \partial^2_{z_\alpha z_\beta} \ln \Theta(K^P - \mathcal{A}_P(P_m))v_\alpha(P_m)v_\beta(P_m). \tag{2.25}$$

This lemma is an immediate corollary of (2.12) and the formula (35) from [11], which expresses the second derivative of the theta-function $\Theta(\mathcal{A}|_P^Q - K)$ in terms of the bidifferential W. □

Using (2.13), we can represent the variation of the theta-functional term from the numerator of (2.11) as follows

$$\partial_{\lambda_m} \ln \Theta(\sum_{\gamma=1}^{g-1} \mathcal{A}_P(R_\gamma) + \mathcal{A}_Q(R_g) + K^P \,|\, \mathbf{B}) =$$

$$\sum_{\gamma=1}^{g} \left[\partial_{\lambda_m} \int_{Q+(g-1)P}^{\sum_{\gamma=1}^{g} R_\gamma} v_\alpha + \partial_{\lambda_m} K_\alpha^P \right] \frac{\partial \ln \Theta}{\partial z_\alpha} + \pi i \sum_{\alpha,\beta=1}^{g} \frac{\partial \ln \Theta}{\partial \mathbf{B}_{\alpha\beta}} v_\alpha(P_m) v_\beta(P_m). \tag{2.26}$$

We have

$$\partial_{\lambda_m} \int_{Q+(g-1)P}^{\sum_{\gamma=1}^{g} R_\gamma} v_\alpha = \frac{1}{2} \int_{Q+(g-1)P}^{\sum_{\gamma=1}^{g} R_\gamma} \partial_m \partial_\lambda \ln E(\lambda, P_m) v_\alpha(P_m)\, d\lambda =$$

$$= \frac{1}{2} \partial_m \ln E(P, P_m) v_\alpha(P_m) - \frac{1}{2} \partial_m \ln E(Q, P_m) v_\alpha(P_m) + o(1) \tag{2.27}$$

as $R_1, \dots, R_g \to P$. Now from (2.26), (2.27), (2.16), the heat equation for the theta-function and the obvious relation

$$\partial_m \ln \Theta(K^P - \mathcal{A}_P(P_m)) \equiv \partial_{x_m} \ln \Theta(\int_{x_m}^{P} \vec{v} + K^P)|_{x_m=0} = -\sum_{\alpha=1}^{g} (\ln \Theta)_{z_\alpha} v_\alpha(P_m)$$

it follows that

$$\lim_{R_1,\dots,R_g \to P} \partial_{\lambda_m} \ln \Theta(\sum_{\gamma=1}^{g-1} \mathcal{A}_P(R_\gamma) + \mathcal{A}_Q(R_g) + K^P \,|\, \mathbf{B}) =$$

$$= -\frac{1}{2} \partial_m \ln \Theta(K^P - \mathcal{A}(P_m)) \partial_m \ln[\mathbf{s}(P_m, Q_0) E^g(P_m, P)] - \frac{1}{4} \sum_{\alpha=1}^{g} \partial_{z_\alpha} \ln \Theta(K^P - \mathcal{A}_P(Q)) \partial_m v_\alpha(P_m)$$

$$+ \frac{1}{4\Theta(K^P - \mathcal{A}_P(Q))} \sum_{\alpha,\beta=1}^{g} \partial^2_{z_\alpha z_\beta} \Theta(K^P - \mathcal{A}_P(Q)) v_\alpha(P_m) v_\beta(P_m)$$

$$-\frac{1}{2} \sum_{\alpha=1}^{g} \partial_{z_\alpha} \ln \Theta(K^P - \mathcal{A}_P(Q)) \partial_m [\ln E(Q, P_m)] v_\alpha(P_m)$$

$$= -\frac{1}{2} \partial_m \ln \Theta(K^P - \mathcal{A}(P_m)) \partial_m \ln[\mathbf{s}(P_m, Q_0) E^g(P_m, P)] + \frac{\partial_m^2 \Theta(K^P - \mathcal{A}(P_m))}{4\Theta(K^P - \mathcal{A}_P(P_m))}$$

$$-\frac{1}{2} \sum_{\alpha=1}^{g} \partial_{z_\alpha} \ln \Theta(K^P - \mathcal{A}_P(Q)) \partial_m [\ln E(Q, P_m)] v_\alpha(P_m) + o(1) \tag{2.28}$$

as $Q \to P_m$. The variation of remaining terms in the right hand side of (2.11) is much easier. One has

$$\lim_{R_1,\dots,R_g \to P} \partial_{\lambda_m} \sum_{\alpha<\beta} \ln E(R_\alpha, R_\beta) = 0\,, \tag{2.29}$$

$$\lim_{R_1,\dots,R_g\to P}\partial_{\lambda_m}\sum_{\alpha=1}^{g}\ln \mathbf{s}(R_\alpha,P)=0 \tag{2.30}$$

$$\lim_{R_1,\dots,R_g\to P}\partial_{\lambda_m}\sum_{\alpha=1}^{g}\ln E(Q,R_\alpha)=-\frac{g}{4}\left(\partial_m\ln\frac{E(Q,P_m)}{E(P,P_m)}\right)^2 \tag{2.31}$$

due to (2.15) and (2.17). Now using (2.11), summing up (2.17), (2.28 - 2.31) and (2.25), cleverly rearranging the terms (as Fay does on p. 59 of [12]) and sending $Q\to P_m$, we get

$$\begin{aligned}\partial_{\lambda_m}\mathcal{C}(P)=&\frac{1}{4}\frac{\partial_m^2\Theta(K^P-\mathcal{A}_P(P_m))}{\Theta(K^P-\mathcal{A}_P(P_m))}-\frac{1}{2}\partial_m\ln\Theta(K^P-\mathcal{A}_P(P_m))\partial_m\ln[\mathbf{s}(P_m,Q_0)E^g(P_m,P)]\\&-\frac{1}{4}\partial_m^2\ln E(P_m,P)+\frac{1}{2}\partial_m\ln\mathbf{s}(P_m,Q_0)\partial_m\ln E(P_m,P)+\frac{2g-1}{4}(\partial_m\ln E(P_m,P))^2\\&-\frac{1}{2}\left[\partial_m\ln E(P_m,Q)\left(\sum_{\alpha=1}^{g}\partial_{z_\alpha}\ln\Theta(K^P-\mathcal{A}_P(Q))v_\alpha(P_m)+\partial_m\ln[\mathbf{s}(P_m,Q_0)E^g(P_m,P)]\right)\right.\\&\left.-\frac{1}{2}\frac{\partial_m^2E(P_m,Q)}{E(P_m,Q)}-\sum_{\alpha,\beta=1}^{g}\partial^2_{z_\alpha z_\beta}\ln\Theta(K^P-\mathcal{A}_P(P_m))v_\alpha(P_m)v_\beta(P_m)\right]_{Q=P_m}.\end{aligned}\tag{2.32}$$

Due to (2.3), one has

$$\lim_{Q\to P_m}\partial_m\ln E(P_m,Q)\left(\sum_{\alpha=1}^{g}\partial_{z_\alpha}\ln\Theta(K^P-\mathcal{A}_P(Q))v_\alpha(P_m)+\partial_m\ln[\mathbf{s}(P_m,Q_0)E^g(P_m,P)]\right)$$

$$=\lim_{x_m\to0}\frac{1}{x_m}\left(\partial_m\ln\frac{\mathbf{s}(P_m,Q_0)E^g(P_m,P)}{\Theta(K^P-\mathcal{A}(P_m))}+\sum_{\alpha,\beta=1}^{g}\partial^2_{z_\alpha z_\beta}\ln\Theta(K^P-\mathcal{A}_P(P_m))v_\alpha(P_m)v_\beta(P_m)x_m+O(x_m^2)\right)$$

$$=\sum_{\alpha,\beta=1}^{g}\partial^2_{z_\alpha z_\beta}\ln\Theta(K^P-\mathcal{A}_P(P_m))v_\alpha(P_m)v_\beta(P_m)\ ,$$

where we made use of the fact that the function

$$R\mapsto\frac{\mathbf{s}(R,Q_0)E^g(R,P)}{\Theta(K^P-\mathcal{A}_P(R))} \tag{2.33}$$

for fixed P is holomorphic and single-valued on $\mathcal{L}$ and, therefore, a constant (thus the first term in the brackets vanishes). Using (2.3) we see that

$$\lim_{Q\to P_m}\frac{\partial_m^2E(P_m,Q)}{E(P_m,Q)}=-\frac{1}{2}S_B(x_m)|_{x_m=0}\ .$$

Thus, the last two lines of (2.32) simplify to $-\frac{1}{8}S_B(x_m)|_{x_m=0}$. Using the R-independence of expression (2.33) once again, we may rewrite the first two lines of (2.32) as

$$\frac{1}{4}\partial_m^2\ln[\mathbf{s}(P_m,Q_0)E(P_m,P)^{g-1}]-\frac{1}{4}(\partial_m\ln[\mathbf{s}(P_m,Q_0)E(P_m,P)^{g-1}])^2,$$

which coincides with $\frac{1}{8}S_{Fay}(x_m)|_{x_m=0}$ due to relation (2.24). Formula (2.18) is proved. □

2.2 Dirichlet integral: variational formulas and holomorphic factorization

For the local parameter near the point at infinity ∞_n in this section we shall use the notation $\zeta_n := 1/\lambda$, which is the same as the parameter x_{M+n} from the introduction.

2.2.1 Definition of regularized Dirichlet integral

Let us cut the branched covering $\mathcal{L}$ into N sheets by a family of contours connecting ramification points P_m; in addition, we dissect it along all a-cycles. On each sheet of the covering $\mathcal{L}$ dissected in this way we can define a real-valued function

$$\varphi(P) = \ln \left| \frac{\omega(P)}{d\pi(P)} \right|^2 \tag{2.34}$$

The difference of values of function φ on different sides of cycle a_α equals $4\pi i(K_\alpha^P - \overline{K_\alpha^P})$. The function φ is singular at all the points of the divisor $\mathcal{D}$ (i.e. ramification points $P_1, \dots, P_M$ and points at infinity $\infty_1, \dots, \infty_N$ of the branch covering $\mathcal{L}$) and at the point P_0. The derivative $\partial_\lambda \varphi$ (where $\lambda = \pi(P)$) is holomorphic outside of the singularities of the function φ and does not change under tracing along the a-cycles.

Lemma 2 *Projective connection (2.8) is related to function φ (2.34) everywhere outside of the divisor $\mathcal{D}$ as follows:*

$$S_{Fay}^{P_0}(\lambda) = \varphi_{\lambda\lambda} - \frac{1}{2}\varphi_\lambda^2 \tag{2.35}$$

The proof of this lemma is a simple standard computation.

In terms of the function φ we define M functions $\varphi^{int}(x_m)$, which are analytic in corresponding neighbourhoods of the ramification points P_m, as follows:

$$e^{\varphi^{int}(x_m)} |dx_m|^2 = e^{\varphi(P)} |d\lambda|^2 \ ; \tag{2.36}$$

in analogy to (2.35) we get $S_{Fay}^{P_0}(x_m) = \varphi^{int}_{x_m x_m} - \frac{1}{2}(\varphi^{int}_{x_m})^2$.

Similarly, in a neighbourhood of any point at infinity ∞_n we define the function $\varphi^\infty(\zeta_n)$ of the local parameter ζ_n by the equality $e^{\varphi^\infty}|d\zeta_n|^2 = e^{\varphi}|d\lambda|^2$. The projective connection $S_{Fay}^{P_0}$ in the parameter ζ_n coincides with $\varphi^\infty_{\zeta_n\zeta_n} - \frac{1}{2}(\varphi^\infty_{\zeta_n})^2$.

Using the interplay between the functions φ, φ^{int} and φ^∞, we find the following asymptotics near the ramification points P_m and the poles ∞_n:

$$|\varphi_\lambda(P)|^2 = \frac{1}{4}|\lambda - \lambda_m|^{-2} + O(|\lambda - \lambda_m|^{-3/2}) \quad \text{as } P \to P_m \tag{2.37}$$

and

$$|\varphi_\lambda(P)|^2 = 4|\lambda|^{-2} + O(|\lambda|^{-3}) \quad \text{as} \quad P \to \infty_n \ . \tag{2.38}$$

At the zero P_0 of the differential ω one gets

$$|\varphi_\lambda(P)|^2 = 4(g-1)^2|\lambda - \lambda_0|^{-2} + O(|\lambda - \lambda_0|^{-1}) \quad \text{as} \quad P \to P_0. \tag{2.39}$$

where $\lambda_0 := \pi(P_0)$.

These asymptotics enable us to introduce the following regularized Dirichlet integral

$$\mathbb{D} = \frac{1}{\pi}\mathrm{reg}\int_{\mathcal{L}} |\varphi_\lambda|^2 \widehat{d\lambda} = \frac{1}{\pi}\lim_{\rho\to 0}\left\{I_\rho + \pi(M+8N+8(g-1)^2)\ln\rho\right\}, \tag{2.40}$$

where $\widehat{d\lambda} = |d\lambda\wedge d\bar\lambda|/2$ and

$$I_\rho = \sum_{n=1}^{N}\int_{\mathcal{L}_\rho^{(n)}} |\varphi_\lambda|^2 \widehat{d\lambda}\,. \tag{2.41}$$

Here $\mathcal{L}_\rho^{(n)}$ is the sub-domain of the n-th sheet of covering $\mathcal{L}$ obtained by cutting off the (small) discs of radius ρ centred at the ramification points and (if applicable) P_0 from the (large) disc $\{\lambda\in\mathcal{L}^{(n)} : |\lambda| < 1/\rho\}$.

2.2.2 Holomorphic factorization of Dirichlet integral

The following theorem shows how to compute the Dirichlet integral (2.40) in terms of the local data at the points of divisor $\mathcal{D}$ and points P_0 and Q_0.

Theorem 3 *The regularized Dirichlet integral admits the following representation:*

$$\mathbb{D} = \ln\left|\frac{\sigma^{4-4g}(P_0,Q_0)\prod_{m=1}^{M}\omega(P_m)}{\prod_{n=1}^{N}\omega^2(\infty_n)}\exp\left\{4\pi i\langle\mathbf{r},K^{P_0}\rangle\right\}\right|^2 - 2M\ln 2, \tag{2.42}$$

where vector $\mathbf{r}$ *has integer components given by*

$$2\pi r_\alpha := Var|_{a_\alpha}\left\{Arg\,\frac{\omega(P)}{d\pi(P)}\right\}\,. \tag{2.43}$$

Proof. Applying the Stokes theorem, we get

$$I_\rho = \frac{1}{2i}\left\{\sum_{m=1}^{M}\oint_{P_m} + \sum_{n=1}^{N}\oint_{\infty_n} + \oint_{P_0} + \sum_{\alpha=1}^{g}\int_{a_\alpha^+\cup a_\alpha^-}\right\}\varphi_\lambda\varphi\, d\lambda, \tag{2.44}$$

Here $\oint_{P_m}$ and $\oint_{P_0}$ are integrals over clock-wise oriented circles of radius ρ around the points P_m and P_0 (it should be noted that each of the points P_m belongs to two sheets simultaneously and, therefore, the integration in $\int_{P_m}$ goes over two circles). The $\oint_{\infty_n}$ denotes the integral over the counter-clock-wise oriented circle of radius $1/\rho$ on the n-th sheet; a_α^+ and a_α^- are different shores of the cycle a_α with the opposite orientation. One has the equality

$$\frac{1}{2i}\int_{a_\alpha^+\cup a_\alpha^-}\varphi_\lambda\varphi\, d\lambda = \pi r_\alpha \ln|\exp 4\pi i K_\alpha^{P_0}|^2\,. \tag{2.45}$$

We note that $\varphi_\lambda = \partial_\lambda\ln(\omega(P)/d\pi(P))$, where the function $\omega(P)/d\pi(P)$ is single-valued on the cycle a_α; since the a-cycles are assumed not to contain the point P_0, function $\omega(P)/d\pi(P)$ does not vanish on a_α.

We have also

$$\frac{1}{2i}\oint_{P_m} = \frac{1}{2i}\oint_{|x_m|=\sqrt{\rho}} \left\{\varphi^{int}_{x_m}\frac{1}{2x_m} - \frac{1}{2x_m^2}\right\}\{\varphi^{int} - 2\ln|x_m| - 2\ln 2\}2x_m\,dx_m =$$

$$= \pi\varphi^{int}(x_m)|_{x_m=0} - 2\pi\ln 2 - \pi\ln\rho + o(1), \tag{2.46}$$

$$\frac{1}{2i}\oint_{\infty_n} = \frac{1}{2i}\oint_{|\lambda|=1/\rho}\left\{-\varphi^{\infty}_{\zeta_n}\lambda^{-2} - \frac{2}{\lambda}\right\}\{\varphi^{\infty} - 4\ln|\lambda|\}\,d\lambda =$$

$$= -2\pi\varphi^{\infty}(\zeta_n)|_{\zeta_n=0} - 8\pi\ln\rho + o(1), \tag{2.47}$$

and

$$\frac{1}{2i}\oint_{P_0} = \frac{1}{2i}\oint_{|\lambda-\lambda_0|=\rho} \ln|\sigma^2(P_0,Q_0)(\lambda-\lambda_0)^{2g-2}\{1+O(\lambda-\lambda_0)\}|^2\times$$

$$\times\left(\frac{2g-2}{\lambda-\lambda_0} + O(1)\right)d\lambda = -\pi\ln|\sigma^{4g-4}(P_0,Q_0)|^2 - 8\pi(g-1)^2\ln\rho + o(1), \tag{2.48}$$

as $\rho\to 0$. These asymptotics together with (2.40) and (2.45) imply (2.42). $\square$

2.2.3 Variational formulas for Dirichlet integral

From now on we assume that the projections $\pi(P_0)$ and $\pi(Q_0)$ of the points P_0 and Q_0 from (2.7) are independent of $\{\lambda_m\}$.

Theorem 4 *The variation of the regularized Dirichlet integral $\mathbb{D}$ (2.40), (2.42) with respect to branch points λ_m is given by values of projective connection (2.8) at the ramification points P_m i.e.*

$$\frac{\partial\mathbb{D}}{\partial\lambda_m} = S^{P_0}_{Fay}(x_m(P))|_{P=P_m}, \quad m=1,\dots,M\,. \tag{2.49}$$

We start from the following

Lemma 3 *On every sheet of the covering $\mathcal{L}$, dissected in addition along all a and b-cycles, the derivatives of function $\varphi(P)$ with respect to λ and λ_m are related as follows:*

$$\varphi_{\lambda_m} + F_m\varphi_\lambda + (F_m)_\lambda = 0\,, \quad m=1,\dots,M, \tag{2.50}$$

where functions $F_m(P)$ are defined on the dissected covering $\mathcal{L}$ as follows:

$$F_m(P) = -\frac{\mathcal{U}(P)_{\lambda_m}}{\mathcal{U}(P)_\lambda}\,,$$

and $\mathcal{U}(P) = \int_{P_0}^P\omega$. Near ramification points and points at infinity the functions F_m have the following asymptotics:

$$F_m(P) = O(|\lambda|^2)\,, \qquad \text{as} \qquad P\to\infty_n\,,$$

$$F_m(P) = \delta_{lm} + o(1)\,, \qquad \text{as} \qquad P\to P_l\,,$$

where δ_{lm} is the Kronecker symbol.

The proof of relation (2.50) can be obtained by direct differentiation (one needs to use the fact that the map $\mathcal{U}$ depends on $\{\lambda_m\}$ holomorphically). The proof of the asymptotical behaviour of the functions F_m essentially repeats Lemma 5 from [18].

Considering the exact differentials $d((F_m)_\lambda\varphi)$, $d(F_m\varphi_\lambda\varphi)$, $d(F_m\varphi_\lambda)$ and making use of (2.50), we get the following

Corollary 1 *The following two 1-forms are exact:*

$$\{(\varphi_\lambda\varphi)_{\lambda_m}d\lambda\} - \{F_m|\varphi_\lambda|^2 d\bar\lambda + (F_m)_\lambda\varphi_{\bar\lambda}d\bar\lambda\ \}\ , \tag{2.51}$$

and

$$\{F_m|\varphi_\lambda|^2d\bar\lambda - (F_m)_\lambda\varphi_\lambda d\lambda + (F_m)_\lambda\varphi_{\bar\lambda}d\bar\lambda\} - \{F_m(2\varphi_{\lambda\lambda} - \varphi_\lambda^2)d\lambda + \varphi\varphi_{\lambda\lambda_m}d\lambda\}\ . \tag{2.52}$$

Proof of theorem 4 (the idea of this proof, including lemma 3 goes back to [31]).

By (2.44) we get

$$\frac{\partial I_\rho}{\partial\lambda_m} = \frac{1}{2i}\left\{\oint_{P_m}[(\varphi_\lambda\varphi)_{\lambda_m} + (\varphi_\lambda\varphi)_\lambda\,]\,d\lambda + \left(\sum_{l\neq m}\oint_{P_l} + \sum_{n=1}^{N}\oint_{\infty_n} + \oint_{P_0}\right)(\varphi_\lambda\varphi)_{\lambda_m}\,d\lambda\right\} +$$

$$+\frac{1}{2i}\sum_{\alpha=1}^{g}\int_{a_\alpha^+\cup a_\alpha^-}(\varphi_\lambda\varphi)_{\lambda_m}\,d\lambda\ . \tag{2.53}$$

(One may assume that the projections of basic cycles $\pi(a_\alpha)$ are independent of $\{\lambda_m\}$.)

Using the holomorphy of $(F_m)_\lambda\varphi_\lambda$ and the relation $(\varphi_\lambda\varphi)_\lambda\,d\lambda = d(\varphi_\lambda\varphi) - \varphi_\lambda\varphi_{\bar\lambda}d\bar\lambda$, we rewrite the r. h. s. of (2.53) as

$$\frac{1}{2i}\Big[-\oint_{P_m}|\varphi_\lambda|^2d\bar\lambda + \left\{\sum_{l=1}^{M}\oint_{P_l} + \sum_{n=1}^{N}\oint_{\infty_n} + \oint_{P_0}\right\}\{F_m|\varphi_\lambda|^2 - (F_m)_\lambda\varphi_\lambda\,d\lambda + (F_m)_\lambda\varphi_{\bar\lambda}\,d\bar\lambda\} -$$

$$-\sum_{\alpha=1}^{g}\left[\int_{a_\alpha^+\cup a_\alpha^-} + \int_{b_\alpha^+\cup b_\alpha^-}\right](F_m)_\lambda\varphi_\lambda\,d\lambda + \sum_{\alpha=1}^{g}\int_{a_\alpha^+\cup a_\alpha^-}(\varphi_\lambda\varphi)_{\lambda_m}\,d\lambda\Big]. \tag{2.54}$$

By means of the asymptotical expansions of the integrands at the ramification points and points ∞_n (the asymptotics from Lemma 3 play here a central role; cf. the proof of Theorem 4 in [18]) one gets the relation

$$\frac{1}{2i}\left\{\sum_{l=1}^{M}\oint_{P_l} + \sum_{n=1}^{N}\oint_{\infty_n}\right\}\{F_m|\varphi_\lambda|^2d\bar\lambda - (F_m)_\lambda\varphi_\lambda d\lambda + (F_m)_\lambda\varphi_{\bar\lambda}\,d\bar\lambda\} =$$

$$= \frac{1}{2i}\oint_{P_m}|\varphi_\lambda|^2\,d\bar\lambda - \frac{3\pi}{4}\sum_{l=1}^{M}\frac{d^2F_m}{(dx_l)^2}\Big|_{x_l=0} + o(1) \tag{2.55}$$

as $\rho\to 0$.

By (2.52) we have

$$\frac{1}{2}\oint_{P_0} F_m|\varphi_\lambda|^2 d\bar{\lambda} - (F_m)_\lambda\varphi_\lambda d\lambda + (F_m)_\lambda\varphi_{\bar\lambda}\, d\bar{\lambda} = \frac{1}{2i}\oint_{P_0} F_m(2\varphi_{\lambda\lambda} - (\varphi_\lambda)^2)d\lambda + o(1). \tag{2.56}$$

The Cauchy theorem, the asymptotical expansions at P_l and ∞_n and relation (2.35) imply that

$$0 = \frac{1}{2i}\left\{\sum_{l=1}^{M}\oint_{P_l} + \sum_{n=1}^{N}\oint_{\infty_n} + \oint_{P_0} + \sum_{\alpha=1}^{g}\left[\int_{a_\alpha^+\cup a_\alpha^-} + \int_{b_\alpha^+\cup b_\alpha^-}\right]\right\} F_m(2\varphi_{\lambda\lambda} - (\varphi_\lambda)^2)\, d\lambda =$$

(2.57)

$$= -\frac{3\pi}{4}\sum_{l=1}^{M}\frac{d^2F_m}{(dx_l)^2}\Big|_{x_l=0} + \pi S_{Fay}^{P_0}(x_m)\Big|_{x_m=0} + \frac{1}{2i}\left\{\oint_{P_0} + \sum_{\alpha=1}^{g}\left[\int_{a_\alpha^+\cup a_\alpha^-} + \int_{b_\alpha^+\cup b_\alpha^-}\right]\right\} F_m(2\varphi_{\lambda\lambda} - (\varphi_\lambda)^2)\, d\lambda + o(1)$$

(cf. [18], Lemma 6). Observe that

$$\int_{a_\alpha^+\cup a_\alpha^-} (\varphi_\lambda\varphi)_{\lambda_m}\, d\lambda = \int_{a_\alpha^+\cup a_\alpha^-} [(2\varphi_{\lambda\lambda} - (\varphi_\lambda)^2)F_m + (F_m)_\lambda\varphi_\lambda]\, d\lambda$$

due to (2.52) and an obvious equality

$$\int_{a_\alpha^+\cup a_\alpha^-} \varphi\varphi_{\lambda\lambda_m}\, d\lambda = 0.$$

Similarly,

$$\int_{b_\alpha^+\cup b_\alpha^-} [(2\varphi_{\lambda\lambda} - (\varphi_\lambda)^2)F_m + (F_m)_\lambda\varphi_\lambda]\, d\lambda = 0,$$

due to the equality

$$\frac{\partial}{\partial\lambda_m}\int_{b_\alpha^+\cup b_\alpha^-} \varphi_\lambda\varphi\, d\lambda = 0.$$

To finish the proof it remains to collect together equations (2.53)-(2.57), and make use of the fact that all $o(1)$ in the above equalities are uniform with respect to $(\lambda_1, \dots, \lambda_M)$ belonging to a compact neighbourhood of the initial point $(\lambda_1^0, \dots, \lambda_M^0)$. $\square$

2.3 Calculation of the tau-function

Theorem 5 *The isomonodromic tau-function of Frobenius manifold structure on the Hurwitz space $H_{g,N}(1, \dots, 1)$ is given by the following expression, which is independent of the choice of the points P_0 and Q_0:*

$$\tau^{-6} = \frac{\{\mathbf{s}(P_0, Q_0)\}^{2-2g} e^{2\pi i\langle\mathbf{r}, K^{P_0}\rangle}}{\mathcal{C}^4(P_0)(d\pi(P_0))^{g-1}}\prod_{k=1}^{M+N}\{\mathbf{s}(D_k, Q_0)\}^{d_k}\{E(D_k, P_0)\}^{(g-1)d_k}\ , \tag{2.58}$$

where the integer vector $\mathbf{r}$ is defined as follows:

$$\mathcal{A}(\mathcal{D}) + 2K^{P_0} + \mathbf{B}\mathbf{r} + \mathbf{s} = 0\ ; \tag{2.59}$$

the initial point of the Abel map coincides with P_0 and all the paths are chosen inside the same fundamental polygon $\hat{\mathcal{L}}$.

Proof. Expression (2.58) is an immediate corollary of theorems 3, 4 and formula (2.18). The only thing one needs to check is the coincidence of the vector $\mathbf{r}$ defined by formula (2.43) with the vector $\mathbf{r}$ defined by (2.59). This coincidence is easy to prove for $g \geq 2$. Namely, we know that (2.58), where the components of the vector $\mathbf{r}$ are given by (2.43), gives the tau-function (1.2). Therefore, as P_0 encircles the basic b-cycle, the tau-function can only gain a $\{\lambda_m\}$-independent factor. Computing the monodromy of expression (2.58) along cycle b_α, we see that this implies (2.59) unless $g-1 \neq 0$. For $g=1$ the relation (2.58) follows from the formula for the tau-function which was found in [18].

Now, since (2.59) is proved, we can show that expression (2.58) is independent of P_0 and Q_0. Simple counting of tensor weight shows that expression (2.58) is a 0-differential with respect to each argument P_0 and Q_0, which is, moreover, free of singularities. Due to relation (2.59) and multiplicative properties of the differentials $\mathcal{C}$ and $\mathbf{s}$ we can also check that it is single-valued on $\mathcal{L}$ with respect to each of these arguments, and, therefore, is independent of both of them.

$\square$

To transform expression (2.58) further to the form (1.7) we shall use the following two lemmas

Lemma 4 *The fundamental domain $\hat{\mathcal{L}}$ of the Riemann surface $\mathcal{L}$ can always be chosen such that $\mathcal{A}(\mathcal{D}) + 2K^P = 0$.*

Proof. For an arbitrary choice of the fundamental domain the vector $\mathcal{A}(\mathcal{D}) + 2K^P$ coincides with 0 on the jacobian of the surface $\mathcal{L}$ i.e. there exist two integer vectors $\mathbf{r}$ and $\mathbf{s}$ such that

$$\mathcal{A}(\mathcal{D}) + 2K^P + \mathbf{B}\mathbf{r} + \mathbf{s} = 0$$

Consider the point $P_1 \in \mathcal{D}$; according to our assumptions this is a simple zero of $d\pi$. By a smooth deformation of a cycle a_α within a given homological class we can stretch it in such a way that the point P_1 crosses this cycle; two possible directions of this crossing correspond to the jump of the component $\mathbf{r}_\alpha$ of the vector $\mathbf{r}$ to $+1$ or -1. Similarly, if we deform a cycle b_α in such a way that it gets crossed by the point P_1, the component $\mathbf{s}_\alpha$ of the vector $\mathbf{s}$ also jumps with $+1$ or -1 depending on the direction of the crossing. Repeating such procedure, we get the fundamental domain where $\mathbf{r} = \mathbf{s} = 0$. $\square$

From the proof it is clear that even a stronger statement is true: the choice of the fundamental domain such that $\mathcal{A}(\mathcal{D}) + 2K^P = 0$ is possible if at least one of the ramification points is simple.

Lemma 5 *Assume that the fundamental domain $\hat{\mathcal{L}}$ is chosen in such a way that*

$$\mathcal{A}(\mathcal{D}) + 2K^P = 0. \tag{2.60}$$

Then for any two points $P, Q \in \mathcal{L}$ the function $\mathbf{s}(P,Q)$ can be written as follows in terms of prime-forms:

$$\mathbf{s}^2(P,Q) = \frac{d\pi(P)}{d\pi(Q)} \prod_{k=1}^{M+N} \left(\frac{E(D_k, Q)}{E(D_k, P)} \right)^{d_k} \tag{2.61}$$

Proof. Consider the expression

$$(d\pi(P))^{1-g} \mathcal{C}^{-2}(P) \prod_{k=1}^{M+N} E^{d_k(g-1)}(D_k, P)\,. \tag{2.62}$$

Summing up the tensor weights of all ingredients of this expression, we see that this is a 0-differential with respect to P. Moreover, it is single-valued under tracing along all the basic cycles (this can be easily checked using multiplicative properties of the prime-forms and $\mathcal{C}(P)$), and does not have either zeros or poles on $\mathcal{L}$ (the poles and zeros induced by the prime-forms are cancelled by the poles and zeros of $d\pi(P)$). Therefore, expression (2.62) is independent of the point P. Taking its ratio at arbitrary two points P and Q and using expression (2.10) of $\mathbf{s}(P,Q)$ in terms of ratio of the differential $\mathcal{C}(P)$ at these two points, we get (2.61). $\square$

Now, choosing in formula (2.58) $P_0 = Q_0$ and expressing $\mathbf{s}(D_k, P_0)$ in terms of the prime-forms using (2.61), we get expression (1.7) for the isomonodromic tau-function of Hurwitz Frobenius manifolds stated in the introduction.

Remark 2 It is natural to expect that once the final expression (1.7) is known, the validity of defining equations (1.2) can be proved by a straightforward computation without using the technique of variation and holomorphic factorization of the Dirichlet integral. Such straightforward proof is indeed possible in the genus zero and genus one cases [20]; however, surprisingly enough, it seems to be more technical than the indirect proof using the technique of Dirichlet integral. Therefore, although we believe that the direct proof exists also in the higher genus case, probably, it does not lead to a significant simplification of the proof given here.

Remark 3 For the stratum $H_{2,g}(1,\dots,1)$, which consists of hyperelliptic Riemann surfaces $\nu^2 = \prod_{j=1}^{2g+2}(\lambda-\lambda_j)$ with $M = 2g+2$ simple branch points, the tau-function τ was computed in [16] in the following form:

$$\tau = \det\mathbf{A} \prod_{j<k,\, j,k=1}^{2g+2} (\lambda_j - \lambda_k)^{1/4} \tag{2.63}$$

where $\mathbf{A}_{\alpha\beta} = \oint_{a_\alpha} \frac{\lambda^{\beta-1}}{\nu}$ is the matrix of a-periods of non-normalized holomorphic abelian differentials on $\mathcal{L}$. To verify that expression (1.7) in hyperelliptic case gives rise to (2.63) we need to use the representation (2.11) for the differential $\mathcal{C}(P)$, together with the formula (2.61) for the differential $\mathbf{s}(P,Q)$ and assume that $g+1$ arbitrary points $Q, R_1, \dots, R_g$ tend to $g+1$ different branch points (say, $\lambda_1, \dots, \lambda_{g+1}$). The theta-function entering (2.11) can be then computed via Thomae formula in terms of $\det\mathbf{A}$ and pairwise differences of the branch points. The $\det||v_\alpha(R_\beta)||$ is also easily represented in the same terms. Collecting all appearing contributions, we arrive at (2.63).

2.3.1 Genus 1 case

The differential $\mathcal{C}(P)$ in elliptic case does not depend on P ([12], p.21):

$$\mathcal{C} = \eta^3(\mathbf{B}) e^{-\pi i \mathbf{B}/4} ,$$

where $\eta(\mathbf{B})$ is the Dedekind eta-function. The differential $\mathbf{s}(P,Q)$ is given by

$$\mathbf{s}(P,Q) = \exp\left\{\pi i \int_Q^P v\right\} \frac{\sqrt{v(P)}}{\sqrt{v(Q)}} \tag{2.64}$$

Substituting these expressions to (2.58) and taking into account (2.59), we get the following expression:

$$\tau = \eta^2(\mathbf{B}) \prod_{k=1}^{M+N} \{v(D_k)\}^{-d_k/12} , \tag{2.65}$$

where according to our usual conventions $v(D_k) := v(P)/dx_k(P)|_{P=D_k}$, $k = 1,\dots,M+N$. This formula was independently proved in [18]; this confirms correctness of the choice of integer $\mathbf{r}$ in (2.58) for $g = 1$.

2.4 Tau-function for an arbitrary stratum of Hurwitz space

Here we briefly consider the general case, when the critical points and poles of function $\pi(P)$ have arbitrary multiplicities. Such tau-function arises for general Hurwitz Frobenius manifolds from ([4]) and in the problem of computation of the subleading term in the large N expansion of the partition function in hermitian two-matrix model [9] (in this case the multiplicities of poles of $\pi(P)$ can be arbitrary, and the branch points are simple). In the problem of computation of isomonodromic tau-function corresponding to Riemann-Hilbert problem with arbitrary permutation monodromies [22] the multiplicities of the critical points can be arbitrary. As before, denote the branch points of the branched covering $\mathcal{L}$ by $P_1,\dots,P_M$ and assume that they have multiplicities $d_1,\dots,d_M$; the orders of the poles $\infty_1,\dots,\infty_L$ of π we denote by $d_{M+1}-1,\dots,d_{M+L}-1$, respectively. One has $N = \sum_{s=1}^{L} d_{M+s}$. Then divisor $\mathcal{D} := (d\pi)$ can be formally written in the same form as before:

$$\mathcal{D} = \sum_{k=1}^{M+L} d_k D_k \tag{2.66}$$

where $D_m := P_m$, $m = 1,\dots,M$ and $D_{M+s} = \infty_s$, $s = 1,\dots,L$. The genus of the Riemann surface $\mathcal{L}$ is given in this case by the formula:

$$g = \frac{1}{2}\sum_{m=1}^{M} d_m - \sum_{s=1}^{L} d_{M+s} + 1 \tag{2.67}$$

The definition (1.2) generalises as follows:

$$\frac{\partial}{\partial\lambda_m}\ln\tau = -\frac{1}{6(d_m-1)!(d_m+1)}\left(\frac{d}{dx_m(P)}\right)^{d_m-1} S_B(x_m(P))\Big|_{P_m=P}\,, \qquad m = 1,\dots,M\,. \tag{2.68}$$

Compatibility of the system (2.68) follows from Schlesinger equations [22]; it was checked directly in [18] using Rauch variational formulas. By using the same technique as in the case of simple branch points and infinities one can verify that the formula (1.7) stated in the introduction remains valid in the general case after substitution of corresponding multiplicities d_k and genus g. We don't present the proof here since it does not contain any new essential ideas in comparison with the case of simple poles and zeros.

3 Applications of the tau-function of Hurwitz Frobenius manifolds

3.1 G-function of Frobenius manifolds

Fix a stratum $H_{g,N}(k_1,\dots,k_L)$ of Hurwitz space, for which all the critical points of function $\pi(P)$ are simple, but infinities have arbitrary multiplicities $k_1,\dots,k_L$ (for simple infinities all $k_s = 1$). Then the divisor $\mathcal{D}$ (2.66) of the differential $d\pi$ looks as follows:

$$\mathcal{D}_{Frob} = \sum_{m=1}^{M} P_m - \sum_{s=1}^{L}(k_s+1)\infty_s \tag{3.1}$$

i.e. $d_m = 1$ for $m = 1, \ldots, M$ and $d_{M+n} = k_n + 1$ for $n = 1, \ldots, L$. The structures of Frobenius manifold on any Hurwitz space of this type were introduced by Dubrovin [4]. We refer to [4] or [23] and recent papers [27, 28] for definition of all ingredients (Frobenius algebra, prepotential, canonical and flat coordinates, Darboux-Egoroff metrics, G-function) of this construction. Here we shall only discuss the G-function (genus one free energy of Dijkgraaf and Witten), which gives a solution of Getzler equation (for classes of Frobenius manifolds related to quantum cohomologies the G-function is the generating function of elliptic Gromov-Witten invariants).

Recall that each Frobenius structure on the Hurwitz space corresponds to a so-called primary differential φ on the covering $\mathcal{L}$. The arising Frobenius manifold will be denoted by M_φ. In [7, 8] it was found the following expression for the G-function of an arbitrary semisimple Frobenius manifold:

$$G = \ln\left(\frac{\tau_I}{J^{\frac{1}{24}}}\right). \tag{3.2}$$

where τ_I is the Dubrovin's tau-function associated to an arbitrary semisimple Frobenius manifold (in [4] τ_I is called the "isomonodromic tau-function", although, as was shown in [19], it is related to the original definition of Jimbo-Miwa, which we follow here, by $\tau_I = \tau^{-1/2}$); J is the Jacobian of the transformation from the flat to the canonical coordinates.

Using the known expression of Jacobian J in terms of diagonal coefficients of Darboux-Egoroff metric (see, e. g., [7]), we get the following

Theorem 6 *The G-function of the Frobenius manifold M_φ can be expressed as follows*

$$G = -\frac{1}{2}\ln\tau - \frac{1}{48}\sum_{m=1}^{M}\ln \operatorname{Res}_{P_m}\frac{\varphi^2}{d\lambda}, \tag{3.3}$$

where τ is the tau-function on the Hurwitz space $H_{g,N}(k_1, \ldots, k_L)$ given by (1.7), with the divisor $\mathcal{D}$ given by (3.1).

3.2 Genus one free energy of hermitian two-matrix model

Another application of the tau-function (1.7) is in the theory of hermitian one- and two-matrix models [9]. Consider the partition function of hermitian two-matrix model

$$e^{-N^2 F} := \int dM_1 dM_2 e^{-N\mathrm{tr}\{V_1(M_1)+V_2(M_2)-M_1M_2\}}. \tag{3.4}$$

where the integration goes over all independent matrix entries of $N\times N$ hermitian matrices M_1 and M_2; V_1 and V_2 are two polynomial potentials (sometimes it is convenient to consider V_1 and V_2 as infinite power series). The expansion $F = \sum_{G=0}^{\infty} N^{-2G} F^G$ as $N \to \infty$ (so-called "genus expansion") plays an important role in the theory, since the coefficients F^G appear both in statistical physics (Ising model) as well as in enumeration of genus G graphs (see for example [5]). If polynomials V_1 and V_2 are of even degree with positive leading coefficients, then asymptotically, as $N \to \infty$, the main contribution to the partition function (3.4) is given by the matrices whose eigenvalues are concentrated in a finite set of intervals. The intervals filled by the eigenvalues of the matrix M_1 lie around the minima of the potential V_1; the eigenvalues of the matrix M_2 fill the intervals around the minima of the potential V_2.

The intervals supporting eigenvalues of matrices M_1 and M_2 correspond to the so-called spectral algebraic curve $\mathcal{L}$, defined by equation

$$(V_1'(x) - y)(V_2'(y) - x) - \mathcal{P}^0(x, y) + 1 = 0 \tag{3.5}$$

where the polynomial of two variables $\mathcal{P}^0(x,y)$ is the zeroth order term in $1/N^2$ expansion of the polynomial

$$\mathcal{P}(x,y) := \frac{1}{N}\Big\langle \mathrm{tr}\frac{V_1'(x)-V_1'(M_1)}{x-M_1}\frac{V_2'(y)-V_2'(M_2)}{y-M_2}\Big\rangle\;; \tag{3.6}$$

(the notation $\langle Q(M_1,M_2)\rangle$ is used to define the expectation value of any functional Q of the matrices M_1 and M_2 with respect to the integration measure in (3.4)). The branch cuts of the spectral curve $\mathcal{L}$ corresponding to projection of $\mathcal{L}$ on the x-plane coincide with the intervals supporting eigenvalues of M_1 in the limit $N\to\infty$; the branch cuts corresponding to projection of $\mathcal{L}$ on y-plane are the intervals supporting the eigenvalues of M_2.

The equations for derivatives of the functions F^G with respect to coefficients of polynomials $V_{1,2}$ arise as a corollary of a reparametrization invariance of the matrix integral (3.4) (the so-called "loop equations"). In particular, the formula for the leading order term F^0 ("genus zero free energy") in terms of standard holomorphic objects associated to the spectral curve $\mathcal{L}$ was derived in [1]. Results of [1], together with [2], show, that F_0 satisfies so-called generalised WDVV equations, together with a quasi-homogeneity equation, thus indicating the existence of a close link between the large N limit of hermitian matrix models and the theory of Frobenius manifolds. Further confirmation of this link was obtained in [9] where it was shown that the genus one contribution F^1 to the free energy is given by the formula

$$F^1 = \frac{1}{2}\ln\tau + \frac{1}{48}\ln\left\{ (v_{d_2+1})^{1-\frac{1}{d_2}}\prod_{m=1}^{M}\mathrm{res}|_{P_m}\frac{(dy)^2}{dx}\right\} + C \tag{3.7}$$

where v_{d_2+1} is the highest order coefficient of the polynomial V_2; $P_1,\dots,P_M$ are zeros of the differential dx on the spectral curve (i.e. the branch points of the spectral curve realized as a covering of the x-plane), which are assumed to be simple; τ is the isomonodromic tau-function of a Hurwitz Frobenius manifold associated to the spectral curve (3.5). The formula (1.7) proved in this paper gives an explicit expression for the genus one free energy (3.7). The one-matrix model appears when the degree of polynomial V_2 equals 2; in this case the spectral curve (3.5) is hyperelliptic.

A surprising similarity of the expression for the G-function (3.3) of Hurwitz Frobenius manifolds with the expression for the genus one free energy (3.7) of Hermitian two-matrix models is an additional evidence of existence of a close link between hermitian matrix models and 2d topological field theories.

3.3 Determinant of Laplacian in Poincaré metric

Here we consider an application of the tau-function (1.7) to computation of the determinants of Laplacians on Riemann surface in the Poincaré metric (in the trivial line bundle); such determinants are defined in terms of the corresponding ζ-functions as follows: $\det\Delta := \exp\{-\zeta_\Delta'(0)\}$. For elliptic case and the flat metric $|v|^2$ (where v is holomorphic normalized differential) this determinant is given by the Ray-Singer formula (see [26] and formula (3.13) below). Such explicit formula is absent for $g>1$ for Poincaré metric, although variational formulas for $\det\Delta$ with respect to the moduli of the Riemann surface are well-known (see, e. g., [32], or [12] (formulae (5.4) and (4.58))). As it was shown in [18], these formulas imply the following expression for the derivative of $\det\Delta$ with respect to a simple branch point of the covering $\mathcal{L}$:

$$\frac{\partial}{\partial\lambda_m}\left\{\frac{\det\Delta}{\det\Im\mathbf{B}}\right\} = -\frac{1}{12}(S_B - S_{Fuchs})(P_m)\;, \tag{3.8}$$

where S_B is the Bergman projective connection, and $S_{Fuchs}(P) := \{z(P), x(P)\}$ is the Fuchsian projective connection on $\mathcal{L}$, where $z(P)$ if the fuchsian uniformization coordinate; $x(P)$ is a local parameter.

Using this variational formula, in [18] it was obtained a formula which expresses $\det\Delta$ in terms of the tau-function of Hurwitz Frobenius manifolds. To formulate the theorem which combines this result with formula (1.7) we need to introduce a few new objects.

Let all the critical points and poles of the function π be simple (this is sufficient for computation of $\det\Delta$ since on any Riemann surface we can find a meromorphic function with these properties). For $g > 1$ the Riemann surface $\mathcal{L}$ is biholomorphically equivalent to the quotient space $\mathbb{H}/\Gamma$, where $\mathbb{H} = \{z \in \mathbb{C} : \Im z > 0\}$; Γ is a strictly hyperbolic Fuchsian group. Denote by $\pi_F : \mathbb{H} \to \mathcal{L}$ the natural projection. Let x be a local parameter on $\mathcal{L}$. Introduce the standard metric of the constant curvature -1 on $\mathcal{L}$:

$$e^{\chi}|dx|^2 = \frac{|dz|^2}{|\Im z|^2}\,, \tag{3.9}$$

where $z \in \mathbb{H}$, $\pi_F(z) = P$, $x = x(P)$.

In complete analogy to constructions of Sec. 2.2.1, introduce the real-valued functions $\chi(\lambda)$, $\chi^{int}(x_m)$, $m = 1, \dots, M$ and $\chi_n^{\infty}(\zeta_n)$, $n = 1, \dots, N$ by specifying the local parameter $x = \lambda$, $x = x_m$ and $x = \zeta_n$ (in a neighbourhood of the point at infinity of the n-th sheet) in (3.9) respectively.

Consider domains $\mathcal{L}_\rho^{(n)}$ of $\mathcal{L}$ as in the integrals (2.41). (Recall that the domain $\mathcal{L}_\rho^{(n)}$ is obtained from the n-th sheet of $\mathcal{L}$ by deleting small discs around ramification points belonging to this sheet, and the disc around infinity.)

Define the regularized Dirichlet integral analogous to (2.40):

$$\mathbb{D}_F := \frac{1}{\pi}\lim_{\rho\to 0}\left(\sum_{n=1}^{N}\int_{\mathcal{L}_\rho^{(n)}}|\partial_\lambda\chi|^2\widehat{d\lambda} + (8N+M)\pi\ln\rho\right). \tag{3.10}$$

Define the function $\mathbb{S}_F$ by

$$\mathbb{S}_F(\lambda_1, \dots, \lambda_M) = -\frac{1}{12}\mathbb{D}_F - \frac{1}{6}\sum_{m=1}^{M}\chi^{int}(x_m)\Big|_{x_m=0} + \frac{1}{3}\sum_{n=1}^{N}\chi_n^{\infty}(\zeta_n)\Big|_{\zeta_n=0}\,; \tag{3.11}$$

Now we are in a position to formulate the following

Theorem 7 *Consider the Hurwitz space $H_{g,N}(1, \dots, 1)$. Let the pair $(\mathcal{L}, \pi)$ belong to $H_{g,N}(1, \dots, 1)$. Then the determinant of the Laplace operator on $\mathcal{L}$ (acting in the trivial line bundle) in Poincaré metric is given by the following expression:*

$$\det\Delta = c_{g,N}\,\{\det\Im\mathbf{B}\}\,e^{\mathbb{S}_F}\,|\tau|^2. \tag{3.12}$$

where $c_{g,N}$ is a constant independent of the point $(\mathcal{L}, \pi) \in H_{g,N}(1, \dots, 1)$; $\mathbf{B}$ is the matrix of b-periods on $\mathcal{L}$; τ is the isomonodromic tau-function of Frobenius structure on $H_{g,N}(1, \dots, 1)$ given by (1.7).

The formula (3.12) can be considered as a natural generalisation of the Ray-Singer formula for the determinant of Laplacian on the torus with flat metric and periods 1 and σ [26]:

$$\det\Delta = C|\Im\sigma|^2|\eta(\sigma)|^4 \tag{3.13}$$

where η is the Dedekind eta-function. The important feature of (3.13) is that the function

$$\frac{\det \Delta}{\{\Im\sigma\}\{Area(\mathcal{L})\}}$$

is represented as the modulus square of a holomorphic function on the moduli space. This is not the case for the higher genus formula (3.12) due to the presence of the factor $e^{\mathbb{S}_F}$, which does not admit the holomorphic factorization, since the second order holomorphic-antiholomorphic derivatives of the logarithm of this function are non-trivial [12, 32].

Actually, more natural higher genus analog of the Ray-Singer formula (3.13) is given by the determinant of Laplacian computed in Strebel metrics (flat metrics with conic singularities), which are given by the modulus of holomorphic quadratic differential (or, in particular, by the modulus square of a holomorphic Abelian differential) [21].

3.4 Riemann-Hilbert problems with quasi-permutation monodromies and isomonodromic tau-function

The Riemann-Hilbert problem of construction of $GL(N)$-valued function on the universal covering of punctured Riemann sphere $\mathbb{C}P^1 \setminus \{\lambda_1, \dots, \lambda_M\}$ with prescribed monodromy representation in general case (for an arbitrary representation) can not be solved in terms of known special functions. For an arbitrary quasi-permutation monodromy group (i.e. such that each monodromy matrix has exactly one non-vanishing entry in each of its columns and each of its raws) the RH problem was solved in [22] outside of a divisor in the space of monodromy data (the so-called Malgrange divisor, or the divisor of zeros of the Jimbo-Miwa tau-function) following previous works [16, 3], where the 2×2 case was solved. One usually requires the solution Ψ of the Riemann-Hilbert problem to be normalized to the unit matrix at some point $\lambda_0 \in \mathbb{C}P^1$, which does not coincide with singularities $\{\lambda_m\}$; we shall denote such normalized solution by $\Psi(\lambda, \lambda_0)$.

Theorem 8 *[22] Let the set of the monodromy data lie outside of the Malgrange divisor. Then the solution $\Psi(\lambda, \lambda_0)$ of an arbitrary Riemann-Hilbert problem with quasi-permutation monodromy representation is given by the analytical continuation on universal covering of the punctured sphere of the following expression defined in a neighbourhood of the normalization point (all objects in this formula correspond to the N-sheeted branched covering $\mathcal{L}$, associated with the quasi-permutation monodromy representation):*

$$\Psi_{kj}(\lambda_0, \lambda) = \frac{\lambda - \lambda_0}{\sqrt{d\lambda d\lambda_0}} \frac{\Theta\left[\begin{smallmatrix}\mathbf{p}\\ \mathbf{q}\end{smallmatrix}\right]\left(\mathcal{A}(\lambda^{(j)}) - \mathcal{A}(\lambda_0^{(k)}) + \Omega\right)}{\Theta\left[\begin{smallmatrix}\mathbf{p}\\ \mathbf{q}\end{smallmatrix}\right](\Omega) E(\lambda^{(j)}, \lambda_0^{(k)})} \prod_{m=1}^{M} \prod_{l=1}^{N} \left[\frac{E(\lambda^{(j)}, \lambda_m^{(l)})}{E(\lambda_0^{(k)}, \lambda_m^{(l)})}\right]^{r_m^{(l)}} \tag{3.14}$$

where

$$\Omega := \sum_{m=1}^{M} \sum_{j=1}^{N} r_m^{(j)} \mathcal{A}(\lambda_m^{(j)})\ ; \tag{3.15}$$

$\lambda^{(k)}$ denotes the point of $\mathcal{L}$ which belongs to the kth sheet and has projection λ on $\mathbb{C}P^1$; $\mathbf{p}, \mathbf{q} \in \mathbb{C}^g$ are constant vectors; $r_m^{(k)}$ are constants assigned to all points from $\pi^{-1}(\lambda_m)$ (if two points from $\pi^{-1}(\lambda_m)$ coincide, the constants $r_m^{(k)}$ are assumed to coincide, too). The logarithms of the matrix elements of monodromy matrices are linear functions of the constants $\mathbf{p}, \mathbf{q}$ and $r_m^{(k)}$. The Malgrange divisor is defined by the equation $\Theta\left[\begin{smallmatrix}\mathbf{p}\\ \mathbf{q}\end{smallmatrix}\right](\Omega) = 0$.

If the elements of monodromy matrices (or, equivalently, the constants $\mathbf{p}, \mathbf{q}$ and $r_m^{(k)}$) are independent of positions of singularities $\{\lambda_m\}$, function Ψ defines a solution of the Schlesinger system, together with isomonodromic tau-function of Jimbo-Miwa [15], defined as follows:

$$\frac{\partial}{\partial \lambda_m} \ln \tau_1 = \frac{1}{2} \mathrm{res}|_{\lambda=\lambda_m} \mathrm{tr} \left(\Psi_\lambda \Psi^{-1}\right)^2 \tag{3.16}$$

In [22] it was proved the following

Theorem 9 *The Jimbo-Miwa tau-function corresponding to solution (3.14) of the Riemann-Hilbert problem, is given by the following formula:*

$$\tau_1 = \tau^{-1/2} \prod_{m,l=1}^{M} (\lambda_m - \lambda_l)^{r_{ml}} \Theta \left[\begin{smallmatrix} \mathbf{p} \\ \mathbf{q} \end{smallmatrix}\right] (\Omega | \mathbf{B}) \tag{3.17}$$

where τ is the tau-function defined by (1.2);

$$r_{mn} = \sum_{k=1}^{N} r_m^{(k)} r_n^{(k)} \; .$$

This theorem, together with expression (1.7) derived in this paper, gives the explicit formula for Jimbo-Miwa tau-function corresponding to general Riemann-Hilbert problem with quasi-permutation monodromies. For monodromy groups corresponding to hyperelliptic curves this tau-function was found in [16]; for Z_N curves with $N > 2$ it was computed in [10].

We notice that the monodromy groups corresponding to fuchsian Riemann-Hilbert problems of Hurwitz Frobenius manifolds, are not known explicitly, in contrast to monodromy groups corresponding to solutions (3.14). Therefore, one of the natural next problems is to find the monodromy group and the solution of the Riemann-Hilbert problem which correspond to the tau-function (1.7).

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